Colouring of Gphs

Proper Colouring → Painting all the vertices of a gph with colours s.t. no two adjacent vertices have the same colour is called the proper colouring of a gph. → Properly coloured gph.

- We ~~want~~ require min. no. of colours for proper colouring.

A gph. G that requires K different colours for its proper colouring and no less is called a K-chromatic gph. and K → chromatic no. of G.

- For colouring problems, we need to consider only simple, connected gphs
- A gph. consisting of only isolated vertices is 1-chromatic
- A gph. with one or more edges is at least 2-chromatic
- A complete gph. with n vertices is n-chromatic.
- A gph. consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.

Note → Every tree with 2 or more vertices is 2-chromatic.

but not every 2-chromatic gph is a tree.

e.g. Utilities gph. is not a tree.

Note → A gph. with at least one edge is 2-chromatic iff it has no circuits of odd length.

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Note → If d_{max} is the max. degree of the vertices in a gph G then → (1a)

Chromatic no. of G $\leq 1 + d_{max}$

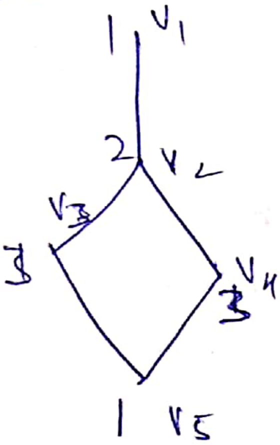
Note →. Every 2-chromatic gph is bipartite coz the colouring partitions the vertex set into 2 disjoint subsets V_1 & V_2 s.t. no two vertices in V_1 (or V_2) are adjacent. Sly every bipartite gph is 2-chromatic

Def^n →. A gph G is called p-partite if its vertex set can be decomposed into p disjoint subsets $V_1, V_2, \ldots V_p$ s.t. no edge in G joins the vertices in the same subset.

Note →. A k-chromatic gph. is p-partite iff $k \leq p$

Chromatic Partitioning →

A proper colouring of a gph. induces a partitioning of the vertices into different subsets. e.g.



Proper colouring

→ $\{v_1, v_5\}, \{v_3, v_4\}, \{v_2\}$

No two vertices in any of these three subsets are adjacent.

→ Such a subset of vertices is called an ind. set.

— A single vertex in any gph. constitutes an ind. set

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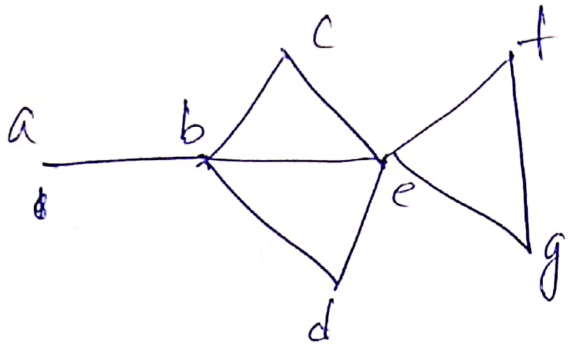
A maximal ind. set is an ind. set to which no other vertex can be added without destroying its independence property. (2)



$\{a, c, d, f\} \rightarrow$ maximal ind. set

$\{b, f\}$, $\{b, g\} \rightarrow$ " " "

- ∴ a gph has many maximal ind. sets.
- The no. of vertices in the largest ind. set of a gph G is called the independence no. $\rightarrow \beta(G)$

For a k-chromatic gph. G of n vertices $\rightarrow$

$$\beta(G) \geq \frac{n}{k}$$

- In above gph, the maximal ind. sets of the gph are —

$\{a, c, d, f\}$, $\{a, c, d, g\}$, $\{b, g\}$, $\{b, f\}$, $\{a, e\}$

∴ $\beta(G) = 4$ (size of the set with maximum no. of vertices)

Chromatic no. of G = min. no. of maximal ind. sets which collectively include all the vertices of G.

For the above gph $\rightarrow$ $\{a, c, d, f\}$, $\{b, g\}$ and $\{a, e\}$ satisfy this condn. ∴ The gph is 3-chromatic.

Chromatic Partitioning: Given a simple, connected gph G. Partition all vertices of G into the smallest possible no. of disjoint ind. sets.

This problem is known as the chromatic partitioning of gphs.

Soln → Enumerate all maximal ind. sets and then selecting the smallest no. of sets that include all vertices of the gph., we just solved the problem.

For above gph, following 4 are some chromatic partitions →

{(a,c,d,f), {b,g}, {e}}

{(a,c,d,g), (b,f), (e)}

{(c,d,f), (b,g), (a,e)}

{(c,d,g), (b,f), (a,e)}

Uniquely colorable Gphs → A gph that has only one chromatic partition is called a uniquely colorable gph

Chromatic Polynomial →

" " $P_n(\lambda)$ of a gph with n vertices gives the no. of ways of properly coloring the gph. using λ or fewer colors.